

Intrinsic Ultracontractivity for Lévy Processes

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Abstract

We prove the intrinsic ultracontractivity for the semigroup generated by a large class of symmetric Lévy processes such that the Lévy measure satisfies some conditions in the neighborhood of 0, killed on exiting a bounded and connected Lipschitz domain.

1 Introduction

Intrinsic ultracontractivity has been studied extensively in recent year in the case of the symmetric diffusions (see e.g. [DS], [B]) and the symmetric α -stable process (see e.g. [CS], [K]). The concept of the intrinsic ultracontractivity for non-symmetric semigroups was introduced in [KS].

If the Lévy measure of symmetric Lévy processes X_t is "uniformly separate" from 0 (see (1)) on truncated cone with vertex in the neighborhood of 0, we prove the intrinsic ultracontractivity for semigroup generated by the killed process on exiting a bounded and connected Lipschitz domain (Theorem 8). In the case if the Lebesgue measure is absolutely continuous with respect to the Lévy measure then we show that the semigroup is intrinsic ultracontractive for any bounded open set (Remark 9).

The paper is organized in the following way. In Section 2 we recall some definitions and prove facts about continuous and strictly positivity of a transition density of process killed on exiting a bounded open set. In Section 3 we prove the intrinsic ultracontractivity.

2 Preliminaries

In \mathbb{R}^d , $d \geq 1$, we consider a symmetric Lévy processes X_t . By ν we denote its (nonzero) Lévy measure and by $p(t, x, y) = p(t, x - y)$ the transition densities of X_t , which are assumed to be continuous for every $t > 0$ and defined for every $x, y \in \mathbb{R}^d$. In addition we assume that there exists a constant $c(\delta)$ such that $p(t, x) \leq c$ for $t > 0$ and $|x| \geq \delta$.

We use the notation $C = C(\alpha, \beta, \gamma, \dots)$ to denote that the constant C depends on $\alpha, \beta, \gamma, \dots$. Usually values of constants may change from line to line, but they are always strictly positive and finite. Sometimes we skip in notation that constants depend on usual quantities (e.g. d, D). Next, we give some definitions. We denote

$$\tau_D = \inf\{t > 0 : X_t \notin D\},$$

$$\eta_D = \inf\{t \geq 0 : X_t \notin D\}.$$

Let D be a bounded connected nonempty open set. In order to study the killed process on exiting of D we construct its transition densities by the classical formula

$$p_D(t, x, y) = p(t, x, y) - r_D(t, x, y),$$

where

$$r_D(t, x, y) = E^x[t > \tau_D; p(t - \tau_D, X_{\tau_D}, y)].$$

The arguments used for Brownian motion (see eg. [CZ]) will prevail in our case and one can easily show that $p_D(t, x, y)$, $t \geq 0$, satisfy the Chapman-Kolmogorov equation (semigroup property). Moreover the transition density $p_D(t, x, y)$ is a symmetric function (x, y) a.s.. With the above assumptions of the transition densities of the (free) process one can actually show that $p_D(t, x, \cdot)$ and $p_D(t, \cdot, x)$ can be chosen as continuous functions on D . The semigroup given by the process X_t killed on exiting of D we denote by P_t^D . We set $G_D(x, y) = \int_0^\infty p_D(t, x, y) dt$ and call the Green function for D .

P_t^D is a strongly continuous semigroup of contractions on $L^2(D)$. Because $p_D(t, x, y)$ is symmetric a.e., we obtain that the operator P_t^D is selfadjoint. For D bounded we get from continuity of $p(t, \cdot)$ that

$$p_D(t, x, y) \leq p(t, x - y) \leq \sup_{x \in B(0, \text{diam}(D))} p(t, x) = C_1(t, D).$$

Therefore P_t^D is Hilbert-Schmidt operator, so it's also compact. So, it's well-known that there exists an orthonormal basis of real-valued eigenfunctions $\{\varphi_n\}_{n=0}^\infty$ with corresponding eigenvalues $\{e^{-\lambda_n t}\}_{n=0}^\infty$ satisfying $0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$, where all φ_n are continuous.

We have that $p_D(t, x, \cdot) \in L^2(D)$ so we can represent this function as

$$p_D(t, x, \cdot) = \sum_{n=0}^{\infty} \langle p_D(t, x, \cdot), \varphi_n \rangle \varphi_n.$$

But $\langle p_D(t, x, \cdot), \varphi_n \rangle = P_t^D \varphi_n(x) = e^{-\lambda_n t} \varphi_n(x)$, so

$$p_D(t, x, y) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y).$$

Now, let us observe that the above series are uniformly convergent, it follows from $|\varphi_n| \leq e^{\lambda_n t/3} C_1(t/3, D)$ and

$$\sum_{n=0}^{\infty} e^{-\lambda_n t/3} = \int_D p_D(t/3, x, x) dx \leq C_1(t/3, D)|D|.$$

Hence, we get that $p_D(t, \cdot, \cdot) \in C(D \times D)$. Therefore $p_D(t, x, y) = p_D(t, y, x)$ for any $t > 0$ and $x, y \in D$.

Next, we show that $p_D(t, \cdot, \cdot)$ is strictly positive on $D \times D$. First, let us observe that for any $x \in D$ we have $p_D(t, x, x) > 0$. Indeed,

$$p_D(t, x, x) = \int_D p_D\left(\frac{t}{2}, x, y\right) p_D\left(\frac{t}{2}, y, x\right) dy = \int_D p_D^2\left(\frac{t}{2}, x, y\right) dy \geq (P^x(\tau_D > t/2))^2 / |D| > 0.$$

Let $K \subset D$ be a compact and connected set. By continuity of $p_D(t, \cdot, \cdot)$ we obtain that for any $x \in K$ there is a radius r_x such that

$$p_D(t, x, y) > 0 \quad \text{for } x, y \in B(x, 2r_x).$$

Because K is compact, there are $x_1, \dots, x_k \in K$ such that $K \subset \bigcup_{i=1}^k B(x_i, r_{x_i})$. Now, we use a fact that K is connected to get from the Chapman-Kolmogorov equation that $p_D(kt, x, y) > 0$ for any $x, y \in K$. Hence we have that $p_D(s, x, y) > 0$ for $s \geq kt$ and $x, y \in K$. Therefore $G_D(x, y) > 0$, first for $x, y \in K$ and next for any $x, y \in D$. This give us that $p_D(t, x, y)$ is strictly positive on D for any $t > 0$. So we obtain that φ_0 is strictly positive on D too.

Lemma 1. *For any $x \in D$ and $t > 0$ we have*

$$p_D(t, x, y) \leq C(t, D) E^x \tau_D E^y \tau_D.$$

Proof. By the Chapman-Kolmogorov equation we obtain for $t > 0$

$$p_D(t, x, y) = \int_D p_D(t/2, x, z) p_D(t/2, z, y) dz \leq C_1(t/2, D) P^x(\tau_D > t/2).$$

Applying again the Chapman-Kolmogorov equation together with the above inequality we get

$$\begin{aligned} p_D(t, x, y) &\leq C_1 P^x(\tau_D > t/4) \int_D p_D(t/2, z, y) dz \\ &= C_1 P^x(\tau_D > t/4) P^y(\tau_D > t/2). \end{aligned}$$

The application of Chebyshev's inequality completes the proof. \square

Definition 2. The semigroup $\{P_t^D\}$ is said to be *intrinsic ultracontractive* if, for any $t > 0$, there exists a constant c_t such that

$$p_D(t, x, y) \leq c_t \varphi_0(x) \varphi_0(y), \quad x, y \in D.$$

Proposition 3. *Let D be a bounded connected nonempty open set. Then $\{P_t^D\}$ is intrinsic ultracontractive if and only if there is a constant C such that $E^x \tau_D \leq C \varphi_0(x)$.*

Proof. Suppose that $\{P_t^D\}$ is intrinsic ultracontractive that is

$$p_D(t, x, y) \leq c_t \varphi_0(x) \varphi_0(y).$$

Because $p_D(t, \cdot, \cdot)$ and $\varphi_0(\cdot)$ are continuous and strictly positive, we have (see Theorem 3.2 in [DS]) that there is \tilde{c}_t such that

$$\tilde{c}_t \varphi_0(x) \varphi_0(y) \leq p_D(t, x, y).$$

If we integrate the above inequality with respect to dt we get

$$C \varphi_0(x) \varphi_0(y) \leq G_D(x, y).$$

And by integrating with respect to dy

$$\tilde{C} \varphi(x) \leq E^x \tau_D.$$

Now, suppose that $E^x \tau_D \leq C \varphi_0(x)$. From Lemma 1 we have

$$p_D(t, x, y) \leq C_t E^x \tau_D E^y \tau_D,$$

what ends the proof. □

3 Main results

We prove intrinsic ultracontractivity for the semigroup P_t^D generated by the symmetric Lévy process, whose a Lévy measure satisfies

$$\forall_{r>0, \gamma \in (0, \pi)} \exists_{\rho>0} \inf_{|y|=\rho; \Gamma_\gamma(y)} \nu(\Gamma_\gamma(y) \cap B(0, r)) > 0, \quad (1)$$

where $\Gamma_\gamma(y)$ is a right circular cone of angle γ at the vertex in y .

Notation and the proof of following theorem is similar as in paper [K]. We assume that D is a bounded and connected Lipschitz domain. That is there exist γ_0 and $R_0 > 0$ and a cone $\Gamma_{\gamma_0} = \{(y, x) : 0 < x, y \in \mathbb{R}^{d-1}, \gamma_0 |y| < x\}$ such that for every $Q \in \partial D$, there is a cone $\Gamma_{\gamma_0}(Q)$ with vertex Q , isometric with Γ_{γ_0} and satisfying $\Gamma_{\gamma_0}(Q) \cap B(Q, R_0) \subset D$. Denote $U(\sigma) = \{x \in D : \delta_D(x) < \sigma\}$, where $\sigma \leq \frac{R_0}{4\sqrt{1+\gamma_0^2}}$. Then for any $x \in U(\sigma)$

there are a point y and a cone $\Gamma_{\gamma_0}(y)$ such that $|y - x| < \sigma(1 + \sqrt{1 + \gamma_0^2}) \leq \frac{R_0}{2}$ and $\Gamma_{\gamma_0}(y) \cap B(x, R_0/2) \subset D \cap U(\sigma)^c$.

We fix $x_0 \in D$ and let $r > 0$ be such that $\overline{B(x_0, 2r)} \subset D$. Denote $K = \overline{B(x_0, r)}$, $L = B(x_0, 2r)$, $M = D \setminus K$ and $N = D \setminus L$. We deal that $r \leq \rho_0$. Define stopping time S_n and T_n

$$\begin{aligned} S_1 &= 0, \\ T_n &= S_n + \eta_M \circ \theta_{S_n}, \\ S_n &= T_{n-1} + \eta_L \circ \theta_{T_{n-1}}. \end{aligned}$$

Now, we prove the following lemma.

Lemma 4. *There exists a constant $c = c(D, x_0)$ such that*

$$P^x(X(\eta_M) \in K) \geq cE^x\eta_M$$

for all $x \in \mathbb{R}^d$.

Proof. From (1) we get existing a constant $\sigma_0 \leq r$ such that

$$\inf_{|y|=\sigma_0(1+\sqrt{1+\gamma_0^2}); \Gamma_{\gamma_0}(y)} \nu(\Gamma_{\gamma_0}(y) \cap B(0, R_0/2)) = C_1.$$

Denote $W = \{x \in D : \delta_D(x) \geq \sigma_0/2\} \setminus B(x_0, r)$.

First, we prove that for $x \in W$, we have

$$P^x(X(\tau_M) \in K) \geq c_1, \tag{2}$$

for some constant $c = c(r, D)$. Let ρ_1 be such that

$$\inf_{|y|=\rho_1; \Gamma_1(y)} \nu(\Gamma_1(y) \cap B(0, r)) = C_2 > 0.$$

Denote $J = D \setminus \overline{B(x_0, r - \rho_1/4)}$. Indeed, from the Ikeda-Watanabe formula we have

$$\begin{aligned} P^x(X(\tau_M) \in K) &\geq P^x(X(\tau_J) \in B(x_0, r - \rho_1/4)) \\ &\geq P^x(X(\tau_J) \in B(x_0, r - \rho_1/2)) \\ &= \int_J G_J(x, y) \nu(B(x_0, r - \rho_1/2) - y) dy \\ &\geq \int_W G_J(x, y) \nu(B(x_0, r - \rho_1/2) - y) dy. \end{aligned}$$

Because $p_J(t, \cdot, \cdot)$ is continuous and positive function on $J \times J$ and $W \times W$ is compact subset of $J \times J$, we get $\inf_{x, y \in W} p_J(t, x, y) > 0$. So,

$$\inf_{x, y \in W} G_J(x, y) \geq \int_0^\infty \inf_{x, y \in W} p_J(t, x, y) dt = c > 0.$$

Besides, we have

$$\inf_{y \in B(x_0, r + \rho_1/2)} \nu((B(x_0, r - \rho_1/2) - y) \geq \inf_{|y|=\rho_1; \Gamma_1(y)} \nu(\Gamma_1(y) \cap B(0, r)) > 0.$$

Therefore

$$P^x(X(\tau_M) \in K) \geq \varepsilon \int_{B(x_0, r + \rho_1/2) \setminus B(x_0, r)} \nu(B(x_0, r - \rho_1/2) - y) dy = C > 0.$$

From (2) and the fact that $E^x \tau_M \leq \tilde{C}$ we obtain the claim of the lemma for $x \in W$.

Now, let $x \in D \setminus (W \cup K) = U(\sigma_0/2)$. Then from Strong Markov Property we get

$$\begin{aligned} P^x(X(\tau_M) \in K) &= E^x(P^{X(\tau_{U(\sigma_0/2)})}(X(\tau_M) \in K)) \geq c_2 E^x(E^{X(\tau_{U(\sigma_0/2)})} \tau_M) \\ &= c_2(E^x \tau_M - E^x \tau_{U(\sigma_0/2)}). \end{aligned}$$

And from (2) we obtain

$$\begin{aligned} P^x(X(\tau_M) \in K) &= E^x(X(\tau_{U(\sigma_0/2)}) \in W \cup K, P^{X(\tau_{U(\sigma_0/2)})}(X(\tau_M) \in K)) \\ &\geq c_1 P^x(X(\tau_{U(\sigma_0/2)}) \in W \cup K). \end{aligned}$$

But

$$\begin{aligned} P^x(X(\tau_{U(\sigma_0/2)}) \in W \cup K) &\geq P^x(X(\tau_{U(\sigma_0/2)}) \in D \cap U(\sigma^0)) \\ &= \int_{U(\sigma_0/2)} G_{U(\sigma_0/2)}(x, y) \nu(D \cap U(\sigma^0) - y) dy \\ &\geq C_1 \int_{U(\sigma_0/2)} G_{U(\sigma_0/2)}(x, y) dy = C_1 E^x \tau_{U(\sigma_0/2)} \end{aligned}$$

Hence

$$\begin{aligned} P^x(X(\tau_M) \in K) &= \left(\frac{1}{2} + \frac{1}{2}\right) P^x(X(\tau_M) \in K) \\ &\geq \frac{c_2}{2}(E^x \tau_M - E^x \tau_{D \setminus (W \cup K)}) + \frac{C_1}{2}(E^x \tau_{D \setminus (W \cup K)}) \\ &\geq \frac{c_2 \wedge C_1}{2} E^x \tau_M. \end{aligned}$$

For $x \in D \setminus$ we have $E^x \tau_M = E^x \eta_M$, and the claim of the lemma for $x \in D^c \cup K$ of course is obvious, so it ends the proof. \square

Lemma 5. *For all $x \in \mathbb{R}^d$ there exists a random variable Z such that for all $n \geq Z$ we have $T_n = \eta_D$ almost surely P^x .*

Proof. We will show that there exists a constant $\beta < 1$ such that $P^x(T_n < \eta_D) \leq \beta^n$ for all $x \in \mathbb{R}^d$ and $n \geq 1$.

Let $R = B(x_0, \text{diam}(D)) \setminus K$ and $\varepsilon = \inf_{x, y \in B(x_0, \text{diam}(D) - \rho_1/2) \setminus B(x_0, 2r)} G_R(x, y)$ then from the Ikeda-Watanabe formula for $x \in N$ we get

$$\begin{aligned} P^x(X(\eta_M) \in D^c) &\geq P^x(X(\eta_R) \in B^c(x_0, \text{diam}(D))) \\ &\geq \int_{B(x_0, \text{diam}(D) - \rho_1/2) \setminus B(x_0, 2r)} G_R(x, y) \nu(B^c(x_0 - y, \text{diam}(D))) dy \\ &\geq \varepsilon \int_{B(0, \text{diam}(D) - \rho_1/2) \setminus B(0, \text{diam}(D) - \rho_1)} \nu(B^c(y, \text{diam}(D))) dy \geq \varepsilon C_2 c = 1 - \beta. \end{aligned}$$

Consequently, for any $x \in \mathbb{R}^d$ and $n \geq 1$ we get

$$\begin{aligned} P^x(T_n < \eta_D, T_{n+1} = \eta_D) &= P^x(T_n < \eta_D, S_{n+1} = \eta_D) + P^x(T_n < \eta_D, S_{n+1} < \eta_D, X(T_{n+1}) \in D^c) \\ &= P^x(T_n < \eta_D, S_{n+1} = \eta_D) + P^x(T_n < \eta_D, X(S_{n+1}) \in N, X(\eta_M) \circ \theta_{S_{n+1}} \in D^c) \\ &= P^x(T_n < \eta_D, S_{n+1} = \eta_D) + E^x(T_n < \eta_D, X(S_{n+1}) \in N, P^{X(S_{n+1})}(X(\eta_M) \in D^c)) \\ &\geq (1 - \beta) P^x(T_n < \eta_D, S_{n+1} = \eta_D) + (1 - \beta) P^x(T_n < \eta_D, S_{n+1} < \eta_D) \\ &= (1 - \beta) P^x(T_n < \eta_D). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} P^x(T_{n+1} < \eta_D) &= P^x(T_n < \eta_D) - P^x(T_n < \eta_D, T_{n+1} = \eta_D) \\ &\leq P^x(T_n < \eta_D) - (1 - \beta) P^x(T_n < \eta_D) = \beta P^x(T_n < \eta_D). \end{aligned}$$

Applying the Borel-Cantelli Lemma ends the proof of lemma. \square

The above lemma allow us to prove similarly as Theorem 8 in [K] the following proposition.

Proposition 6. *Let C be an nonempty open subset of D . Then there is c such that*

$$E^x \int_0^{\tau_D} \mathbf{1}_C(X_t) dt \geq c E^x \tau_D.$$

Theorem 7. *There exists a constant C such that*

$$E^x \tau_D \leq C \varphi_0(x),$$

for all $x \in D$.

Proof. We have, for all $t > 0$,

$$e^{-\lambda_0 t} \varphi_0(x) = \int_D p_D(t, x, y) \varphi_0(y) dy.$$

By integration this with respect dt we get

$$\varphi_0(x) = \lambda_0 \int_D G_D(x, y) \varphi_0(y) dy.$$

Because φ_0 is continuous and positive, we obtain that there is a constant $\varepsilon > 0$ such that a set

$$C = \{x : \varphi_0(x) > \varepsilon\}$$

is nonempty. By Proposition 6 we have

$$\begin{aligned} E^x \tau_D &\leq c^{-1} \int_C G_D(x, y) dy \leq (c\varepsilon)^{-1} \int_C G_D(x, y) \varphi_0(y) dy \\ &\leq (c\varepsilon)^{-1} \int_D G_D(x, y) \varphi_0(y) dy = (c\varepsilon\lambda_0)^{-1} \varphi_0(x). \end{aligned}$$

□

Applying Lemma 3 give us the theorem below.

Theorem 8. *Let D be an bounded and connected Lipschitz domain. If the Lévy measure of symmetric Lévy process X_t satisfies (1), then the semigroup $\{P_t^D\}$ is intrinsic ultracontractive.*

Remark 9. Suppose that the symmetric Lévy process X_t has the Lévy measure such that the Lebesgue measure is absolutely continuous with respect to it. Then the semigroup P_t^D is intrinsic ultracontractive for any bounded open set.

Proof. Proof of this remark is the same as the proof of Theorem 1 in [K].

□

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